

# THE NATURAL FREQUENCIES AND MODE SHAPES OF AN EULER-BERNOULLI BEAM WITH A RECTANGULAR CROSS- SECTION WHICH HAS A SURFACE CRACK

L.Emir Sakman<sup>1</sup>, Aşkın Mutlu<sup>1</sup>

Faculty of Engineering – Department of Mechanical Engineering, Istanbul University, Turkey<sup>1</sup>  
sakman@istanbul.edu.tr, askin@istanbul.edu.tr

**Abstract:** The natural frequencies and mode shapes of an Euler-Bernoulli beam with a rectangular cross-section, which has a surface crack, is investigated. The crack is modeled as a change (sudden or gradual) in the cross-section of the beam, and the perturbation approach is used assuming that the crack is much smaller than the beam cross section. Computations of natural frequencies and mode shapes were carried out for four different crack shapes with rectangular, triangular and parabolic profiles when viewed through the side of the beam. The results are listed in non-dimensional form for various values of the parameters characterizing the crack.

**Keywords:** EULER BERNOULLI, BEAM, CRACK

## 1. Introduction

Identification and characterization of cracks in engineering structures is an important problem both theoretically and technically. One type of problem deals with determination of vibration characteristics of the structure with a crack. Another problem might be the characterization of the crack from measured vibration characteristics of the structure; this is usually called an inverse problem. The practical solution of the inverse problem normally involves solving the forward problem for a wide range of crack types. Therefore, the ability to solve the vibration characteristics of a cracked structure is very important.

Most studies dealing with cracked structures in the literature models the crack as a change in the elastic characteristics of the structure, and utilize some type of numerical method. The present study aims to model the detailed geometry of the crack therefore shedding light on the solution of the inverse problem. Furthermore the perturbation theory approach will be used making the solution analytical.

One of the earlier studies, Adams et al. [1], demonstrated a vibration method for non-destructive testing of structures. Shen and Pier [2] studied the convergence of Galerkin approach for beams with symmetric cracks. Papaconomou and Dimarogonas [3] describe a transfer matrix model for a cracked prismatic bar. Chondros and Dimarogonas [4] used a variational formulation to analyze lumped and continuous cracks. Khiem and Toan [5] used a modification of the Rayleigh quotient method for detection of an unknown number of multiple cracks on beams. Saez et al. [6] performed damage detection by solving the inverse problem. Chaudhari and Maiti [7] used a rotational spring to represent the crack and worked on solving the inverse problem based on the measurement of natural frequencies. He and Lin [8] uses an acoustic system for contact-type cracks. Nejad et al. [9] worked on analytical estimation of natural frequencies and mode shapes of a beam having two cracks. Mazanoglu et al. [10] modified the energy-based method presented by Yang et al. [11], to solve the vibration of non-uniform Euler-Bernoulli beams with multiple cracks by defining the crack as a spring. Open edge cracks were investigated by Aydın [12] again modeling the crack as a spring. Caddemi and Morassi [13] proposed a justification of the rotational elastic spring model of an open crack in a beam. Finite element method is very popular for investigation of free vibration analysis of cracked beams [14-17]. Cracks in reinforced concrete structures is another popular research topic [18,19] using Euler-Bernoulli Beam theory. Timoshenko Beam model was also used [20, 21].

The studies mentioned above do not consider the shape of the crack, but model it as a spring or a change in the local elastic properties. In this study the crack is expressed as a change in the beam cross-section as explained in Section 2. This allows the determination of vibration characteristics depending on the shape of the crack.

## 2. Perturbation Method and Governing Equations

Considering an Euler-Bernoulli beam of length L and cross-sectional area A(x), the governing equation for the vibrations is

$$-\frac{\partial^2}{\partial x^2} (EI(x) \frac{\partial^2 y}{\partial x^2}) = m(x) \frac{\partial^2 y}{\partial t^2} \quad (1)$$

where the cross-sectional area of the beam, A(x), varies with coordinate x along the beam. y(x,t) is the displacement, E is the elastic modulus of the beam material, I(x) is the second area-moment of cross section, m(x) is the mass per unit length. The cross section is assumed to be a rectangle with width b(x), and height h(x) changing along the beam; thus

$$A(x) = b(x)h(x) \quad (2a)$$

$$I(x) = \frac{1}{12} b(x)h(x)^3 \quad (2b)$$

$$m(x) = \rho b(x)h(x) \quad (2c)$$

where  $\rho$  is the density of the beam material. Both ends of the beam are assumed to be simply-supported,

$$y = \frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = 0, L \quad (3)$$

The non-damaged beam has uniform cross section with constant width and height  $b_0$  and  $h_0$  respectively. The crack is modeled as a change in the beam cross-sectional dimensions in the form

$$b(x) = b_0 + \varepsilon f(x) \quad (4a)$$

$$h(x) = h_0 + \varepsilon g(x) \quad (4b)$$

where  $\varepsilon$  is a small non-dimensional perturbation parameter and the functions f(x) and g(x) determine the shape of the crack; these are completely general at this point. Substituting (4) into (2) and (1), the vibration equation becomes, ignoring higher order terms,

$$\begin{aligned} & -\frac{\partial^2}{\partial x^2} \left( \frac{E}{12} (b_0 h_0^3 + \varepsilon (3b_0 h_0^2 g(x) + h_0^3 f(x))) \frac{\partial^2 y}{\partial x^2} \right) \\ & = \rho (b_0 h_0 + \varepsilon (b_0 g(x) + h_0 f(x))) \frac{\partial^2 y}{\partial t^2} \end{aligned} \quad (5)$$

The governing equation is non-dimensionalized with the following definitions, starred symbols showing non-dimensional variables,

$$x^* = \frac{x}{L} \tag{6a}$$

$$t^* = \frac{t}{\sqrt{\rho \frac{12L^4}{Eh_0^4}}} \tag{6b}$$

$$y^* = \frac{y}{h_0} \tag{6c}$$

$$G^*(x^*) = \frac{g(x)}{h_0} \tag{6d}$$

$$F^*(x^*) = \frac{f(x)}{b_0} \tag{6e}$$

The non-dimensional vibration equation becomes, omitting stars after this point,

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = -\varepsilon \left[ \begin{array}{l} \frac{\partial^2}{\partial x^2} \left( (3G(x) + (x)) \frac{\partial^2 y}{\partial x^2} \right) \\ + (G(x) + F(x)) \frac{\partial^2 y}{\partial t^2} \end{array} \right] \tag{7}$$

and the boundary conditions

$$y = \frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = 0, 1 \tag{8}$$

Assuming a separated solution of the form

$$y(x, t) = u(x)p(x) \tag{9}$$

leads to

$$\ddot{p} + \lambda^2 p = 0 \tag{10a}$$

$$u'''' - \lambda^2 u + \varepsilon \varphi(u, \lambda) = 0 \tag{10b}$$

Prime and dot denote differentiation with respect to x and t, respectively, and we defined

$$\varphi(u, \lambda) = ((3G(x) + F(x))u'')'' - \lambda^2 u(G(x) + F(x)) \tag{11}$$

Solution of Eq. (10b) is assumed to be in the form of a perturbation series for both the mode shape and the eigenvalue

$$u(x) = u_0(x) + \varepsilon u_1(x) + \dots \tag{12a}$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \dots \tag{12b}$$

Substituting, the zero and first order problems become

$$u_0'''' - \lambda_0^2 u_0 = 0 \tag{13a}$$

$$u_0(0) = u_0''(0) = u_0(1) = u_0''(1) = 0 \tag{13b}$$

and

$$u_1'''' - \lambda_0^2 u_1 - 2\lambda_0 \lambda_1 u_0 + \varphi(u_0, \lambda_0) = 0 \tag{14a}$$

$$u_1(0) = u_1''(0) = u_1(1) = u_1''(1) = 0 \tag{14b}$$

The solution of the zero order problem is

$$u_0 = \sin n\pi x, \lambda_0 = n^2 \pi^2, n = 1, 2, 3, \dots \tag{15}$$

Thus, the first order problem becomes

$$u_1'''' - n^4 \pi^4 u_1 = 2n^2 \pi^2 \lambda_1 \sin n\pi x - \varphi(\sin n\pi x, n^2 \pi^2) \tag{16}$$

The solution of (16) can be written by variation of constants as

$$u_1 = C_1(x)e^{n\pi x} + C_2(x)e^{-n\pi x} + C_3(x) \cos n\pi x + C_4(x) \sin n\pi x \tag{17}$$

Where

$$C_1'(x) = \frac{e^{-n\pi x} (2n^2 \pi^2 \lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2 \pi^2))}{4n^3 \pi^3} \tag{18a}$$

$$C_2'(x) = \frac{e^{n\pi x} (2n^2 \pi^2 \lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2 \pi^2))}{4n^3 \pi^3} \tag{18b}$$

$$C_3'(x) = \frac{\sin n\pi x (2n^2 \pi^2 \lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2 \pi^2))}{2n^3 \pi^3} \tag{18c}$$

$$C_4'(x) = -\frac{\cos n\pi x (2n^2 \pi^2 \lambda_1 \sin n\pi x + \varphi(\sin n\pi x, n^2 \pi^2))}{2n^3 \pi^3} \tag{18d}$$

Integrating (18) and substituting in (17) gives the general solution in the form

$$u_1 = K_1 e^{n\pi x} + K_2 e^{-n\pi x} + K_3 \cos n\pi x + K_4 \sin n\pi x + \int_0^x \left[ C_1(\xi)e^{n\pi\xi} + C_2(\xi)e^{-n\pi\xi} + C_3(\xi) \cos n\pi\xi + C_4(\xi) \sin n\pi\xi \right] \Psi(\xi) d\xi \tag{19}$$

where we defined

$$\Psi(x) = n^2 \pi^2 \lambda_1 \sin n\pi x - \varphi(\sin n\pi x, n^2 \pi^2) \tag{20}$$

for brevity. K's are arbitrary constants found by applying the boundary conditions (14b); the result is the system of equations

$$K_1 + K_2 + K_3 = 0 \tag{21a}$$

$$K_1 + K_2 - K_3 + \frac{1}{2n^4 \pi^4} \left[ (-1)^n - \frac{1}{n\pi} \right] \psi(0) = 0 \tag{21b}$$

$$K_1 e^{n\pi} + K_2 e^{-n\pi} + K_3 (-1)^n + \frac{e^{n\pi}}{4n^3 \pi^3} \int_0^1 e^{n\pi\xi} \psi(\xi) d\xi \tag{21c}$$

$$- \frac{e^{-n\pi}}{4n^3 \pi^3} \int_0^1 e^{-n\pi\xi} \psi(\xi) d\xi + \frac{(-1)^n}{2n^3 \pi^3} \int_0^1 \sin n\pi\xi \psi(\xi) d\xi = 0$$

$$K_1 e^{n\pi} + K_2 e^{-n\pi} - K_3 (-1)^n + \frac{e^{n\pi}}{4n^3 \pi^3} \int_0^1 e^{-n\pi\xi} \psi(\xi) d\xi \tag{21d}$$

$$- \frac{e^{-n\pi}}{4n^3 \pi^3} \int_0^1 e^{n\pi\xi} \psi(\xi) d\xi - \frac{(-1)^n}{2n^3 \pi^3} \int_0^1 \sin n\pi\xi \psi(\xi) d\xi = 0$$

Note that  $K_2$  does not show in these equations; this is due to the fact that the term associated with  $K_2$  in the general solution, Eq. (19), is  $\sin \pi x$ , and that is the solution of the zero order problem. This part,  $K_2 \sin \pi x$ , is therefore discarded and seen as part of the zero order solution. The unknowns in Eqs.(21) are  $K_1, K_2, K_3$  and  $\lambda_1$  (inside  $\psi$ ).

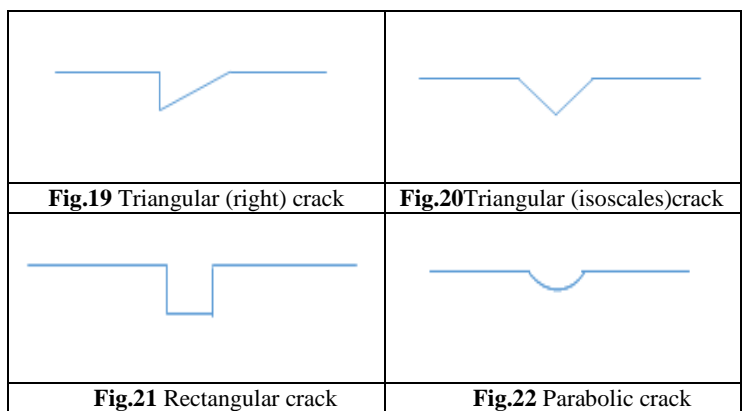
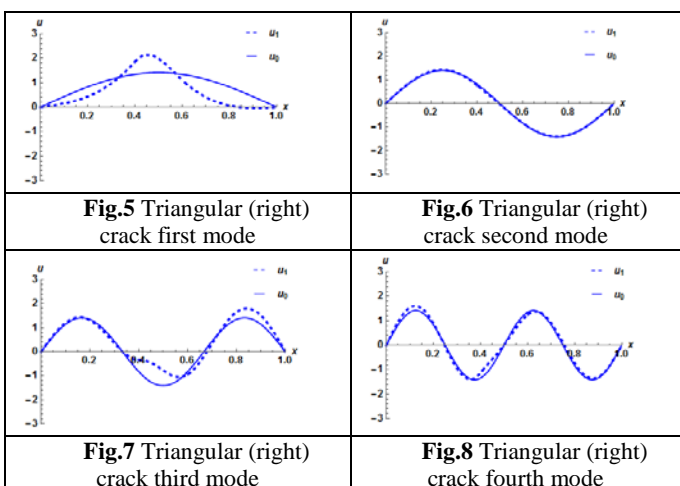
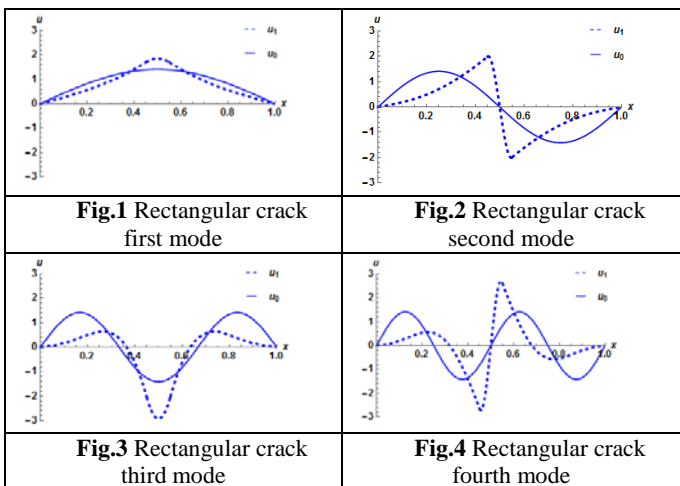
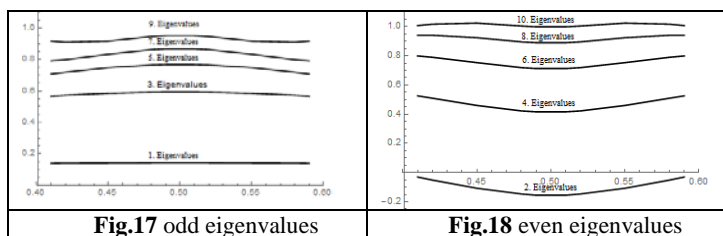
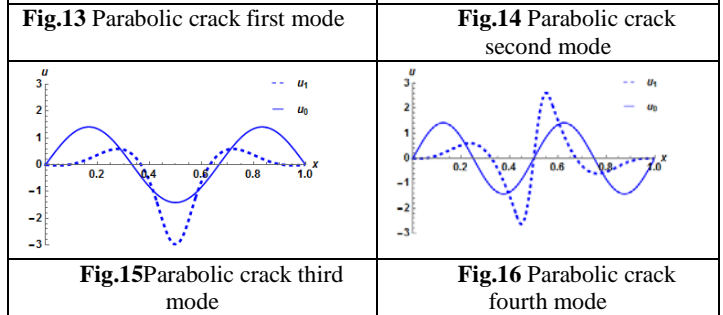
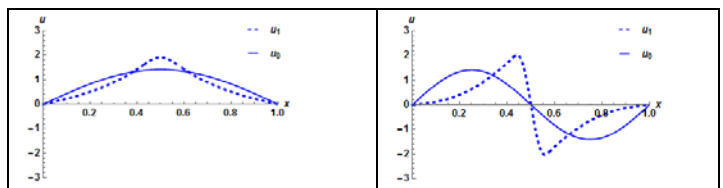
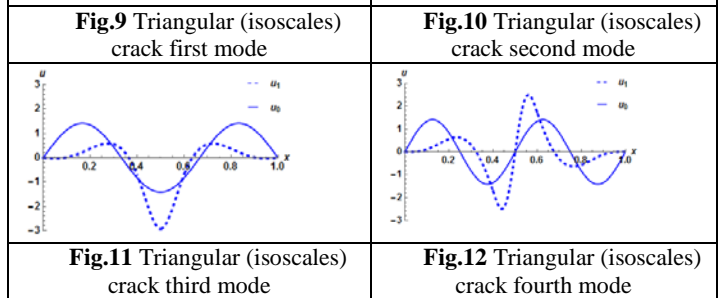
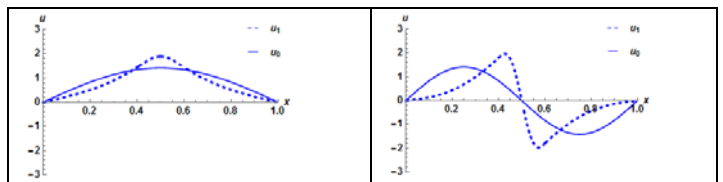
In evaluating the terms in Eqs.(21) we come across expressions like  $F(0), F'(1), G(1)$ , etc. If we assume that the crack is away from the ends ( $x = 0$  and  $1$ ), all these terms become zero.

### 3. Conclusion

The results are listed in non-dimensional form for various values of the parameters characterizing the crack.

Table1: Crack Shape and  $\lambda$

Crack Shape	$\lambda_1^{(1)}$	$\lambda_1^{(2)}$	$\lambda_1^{(3)}$	$\lambda_1^{(4)}$	$\lambda_1^{(5)}$	$\lambda_1^{(6)}$	$\lambda_1^{(7)}$	$\lambda_1^{(8)}$	$\lambda_1^{(9)}$	$\lambda_1^{(10)}$
Rectangular Crack	-1.95	-0.25	-16.50	-3.84	-40.38	-17.60	-66.15	-48.39	-88.68	-98.69
Triangle (Isosceles) Crack	-1.94	-0.49	-15.42	-6.74	-34.67	-26.48	-54.90	-59.71	-80.89	-98.69
Triangle (Right) Crack	-1.91	-0.96	-13.36	-12.09	-24.67	-41.07	-37.90	-75.11	-71.63	-98.69
Parabolic Crack	-1.95	-0.34	-16.11	-4.94	-38.10	-21.39	-60.89	-54.45	-83.19	-101.52



## Nomenclature

$A(x)$	area of cross section
$b(x), h(x)$	width and height of beam cross section
$b_0, h_0$	width and height of non-cracked beam cross section
$I(x)$	second area moment
$m(x)$	mass per unit length
$E$	modulus of elasticity
$\rho$	density
$L$	length of the beam
$\varepsilon$	perturbation parameter
$f(x), g(x)$	crack shape functions
$y(x, t)$	vertical displacement
$u(x)$	mode shape
$p(t)$	separated time function
$\lambda$	eigenvalue
$[ ]^*$	non-dimensional [ ]

## 4. References

- [1] Adams RD, Cawley P, Pye CJ, Stone BJ. A vibration technique for non-destructively assessing the integrity of structures. *J Mech Eng Sci* 1978;20(2):93-100.
- [2] Shen MHH, Pierre C. Natural modes of Bernoulli-Euler beams with symmetric cracks. *J Sound Vib* 1990;138(1):115-134.
- [3] Papaconomou N, Dimarogonas A. Vibration of cracked beams. *Comput Mech* 1989;5:88-94.
- [4] Chondros TG, Dimarogonas AD. Vibration of a cracked cantilever beam. *J Vib Acoust* 1998;120:742-746.
- [5] Khiem NT, Toan LK. A novel method for crack detection in beam-like structures by measurements of natural frequencies. *J Sound Vib* 2014;333:4084-4103
- [6] Sáez JF, Morassi A, Pressacco M, Rubio L. Unique determination of a single crack in a uniform simply supported beam in bending vibration. *J Sound Vib* 2016;371:94-109.
- [7] Chaudhari TD, Maiti SK. Modelling of transverse vibration of beam of linearly variable depth with edge crack. *Eng Fract Mech* 1999;63:425-445.
- [8] He Q, Lin Y. Assessing the severity of fatigue crack using acoustics modulated by hysteretic vibration for a cantilever beam. *J Sound Vib* 2016;370:306-318.
- [9] Nejad FB, Khorram A, Rezaeian M. Analytical estimation of natural frequencies and mode shapes of a beam having two cracks. *Int J Mech Sci* 2014;78:193-202.
- [10] Mazanoglu K, Yesilyurt I, Sabuncu M. Vibration analysis of multiple-cracked non-uniform beams. *J Sound Vib* 2009;320:977-989.
- [11] Yang XF, Swamidas ASJ, Seshadri R. Crack identification in vibrating beams using the energy method. *J Sound Vib* 2001;244(2):339-357.
- [12] Kamil Aydin Free vibration of functionally graded beams with arbitrary number of surface cracks. *Eur J Mech A-Solid* 2013;42:112-124.
- [13] Caddemi S, Morassi A. Multi-cracked Euler-Bernoulli beams: Mathematical modeling and exact solutions. *Int J Solids Struct* 2013;50:944-956.
- [14] Kisa M, Brandon J, Topcu M. Free vibration analysis of cracked beams by a combination of finite elements and component mode synthesis methods. *Comput Struct* 1998;67:215-223.
- [15] Zheng DY, Kessissoglou NJ. Free vibration analysis of a cracked beam by finite element method. *J Sound Vib* 2004;273:457-475.
- [16] Kisa M, Gurel MA. Free vibration analysis of uniform and stepped cracked beams with circular cross sections. *Int J Eng Sci* 2007;45:364-380.
- [17] Blasques JP, Bitsche RD. An efficient and accurate method for computation of energy release rates in beam structures with longitudinal cracks. *Eng Fract Mech* 2015;133:56-69.
- [18] Dundar C, Kara IF. Three dimensional analysis of reinforced concrete frames with cracked beam and column elements. *Eng Struct* 2007;29:2262-2273.
- [19] Xu T, Castel A. Modeling the dynamic stiffness of cracked reinforced concrete beams under low-amplitude vibration loads. *J Sound Vib* 2016;368:135-147.
- [20] Heydari M, Ebrahimi A, Behzad M. Forced vibration analysis of a Timoshenko cracked beam using a continuous model for the crack. *Eng Sci Technol Int J* 2014;17:194-204.
- [21] Panigrahi B, Pohit G. Nonlinear modelling and dynamic analysis of cracked Timoshenko functionally graded beams based on neutral surface approach. *J Mech Eng Sci* 2015;0(0):1-12.